In words, the uncertainty that remains about the value of β_i, summarized by the posterior distribution of $\tilde{\beta}_i$, is such that we assign probability .025 to the possibility that β_i is less than $b_i - 1.96\, s(b_i | R_{m1}, \ldots, R_{mT})$. Likewise,

$$\Pr\left[\tilde{t} = \frac{\tilde{\beta}_i - b_i}{s(b_i | R_{m1}, \ldots, R_{mT})} > 1.96\right] = .025,$$

or

$$\Pr[\tilde{\beta}_i > b_i + 1.96 s(b_i | R_{m1}, \ldots, R_{mT})] = .025, \tag{62}$$

so that we assign probability .025 to the possibility that β_i is greater than $b_i + 1.96 s(b_i | R_{m1}, \ldots, R_{mT})$. Finally, combining equations (61) and (62), the posterior uncertainty about β_i is such that we assign probability .95 to the possibility that β_i is in the interval

$$[b_i - 1.96 s(b_i | R_{m1}, \ldots, R_{mT})] \text{ to } [b_i + 1.96 s(b_i | R_{m1}, \ldots, R_{mT})]. \tag{63}$$

Expression (63) is numerically identical to the classical .95 confidence interval of (58) when applied to a specific sample. When one takes the Bayesian viewpoint, however, the probability statement made from the sample confidence interval is more direct than in the classical approach. In the Bayesian approach, there is no talk about a hypothetical sample or repeated samples. One simply says that the probability that β_i is in the interval given by (63) is .95.

If there is some special interest in a particular possible value of β_i, then the Bayesian can use the posterior distribution of $\tilde{\beta}_i$ to make a direct probability statement about the value of interest. For example, if for IBM there is special interest in the possibility that $\beta_i = 1.0$, the Bayesian computes

$$t = \frac{\beta_i - b_i}{s(b_i | R_{m1}, \ldots, R_{mT})} = \frac{1.0 - .67}{.13} = 2.54. \tag{64}$$

From Table 1.8 he then determines that the posterior probability that β_i is greater than 1.0 is about .005. In other words, the posterior distribution assigns low probability to the possibility that β_i is as large as 1.0. Again, unlike the classical approach, the probability statement is made without reference to repeated samples.

VI. Conclusions

We have spent much time discussing the market model and its estimation from a theoretical viewpoint. We pass on now to the data.

CHAPTER

4

The Market Model: Estimates

The first step in applying the estimation techniques of Chapter 3 to stock market data is to give a detailed example for an individual stock. Then summary results for two samples of 30 stocks are presented, after which we examine some of the practical problems associated with fitting the market model.

I. Estimating the Market Model: A Detailed Example

A. *The Market Model: Summary of Equations and Properties*

It is helpful at this point to summarize the market model equations. Bivariate normality of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ implies that the regression function of $\tilde{R}_{it}$ on $\tilde{R}_{mt}$, the expected value of $\tilde{R}_{it}$ conditional on R_{mt}, is

$$E(\tilde{R}_{it} | R_{mt}) = \alpha_i + \beta_i R_{mt}, \quad t = 1, \ldots, T, \tag{1}$$

with

$$\beta_i = \frac{\operatorname{cov}(\tilde{R}_{it}, \tilde{R}_{mt})}{\sigma^2(\tilde{R}_{mt})} \text{ and } \alpha_i = E(\tilde{R}_{it}) - \beta_i E(\tilde{R}_{mt}), \quad t = 1, \ldots, T. \tag{2}$$

The relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ implied by bivariate normality can be described as

$$\tilde{R}_{it} = \alpha_i + \beta_i \tilde{R}_{mt} + \tilde{\epsilon}_{it}, \quad t = 1, \ldots, T, \tag{3}$$

where the disturbance $\tilde{\epsilon}_{it}$ has mean zero and is independent of $\tilde{R}_{mt}$, so that

$$E(\tilde{\epsilon}_{it} | R_{mt}) = E(\tilde{\epsilon}_{it}) = 0.0, \quad t = 1, \ldots, T, \tag{4}$$

$$\sigma^2(\tilde{R}_{it} | R_{mt}) = \sigma^2(\tilde{\epsilon}_{it} | R_{mt}) = \sigma^2(\tilde{\epsilon}_{it}) = \sigma^2(\tilde{\epsilon}_i), \quad t = 1, \ldots, T, \tag{5}$$

$$\text{cov}(\tilde{\epsilon}_{it}, \tilde{R}_{mt}) = 0.0, \quad t = 1, \ldots, T. \tag{6}$$

It is also helpful to restate the properties of the correlation coefficient ρ_{im} between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ that are described in Chapter 3. Thus,

$$\rho_{im} = \frac{\text{cov}(\tilde{R}_{it}, \tilde{R}_{mt})}{\sigma(\tilde{R}_{it})\, \sigma(\tilde{R}_{mt})}, \quad t = 1, \ldots, T \tag{7}$$

and

$$\sigma^2(\tilde{\epsilon}_{it}) = \sigma^2(\tilde{R}_{it})(1 - \rho_{im}^2), \quad t = 1, \ldots, T, \tag{8}$$

so that

$$\rho_{im}^2 = \frac{\sigma^2(\tilde{R}_{it}) - \sigma^2(\tilde{\epsilon}_{it})}{\sigma^2(\tilde{R}_{it})}, \quad t = 1, \ldots, T. \tag{9}$$

Since the independence of $\tilde{\epsilon}_{it}$ and $\tilde{R}_{mt}$ implies

$$\sigma^2(\tilde{R}_{it}) = \beta_i^2 \sigma^2(\tilde{R}_{mt}) + \sigma^2(\tilde{\epsilon}_{it}), \quad t = 1, \ldots, T, \tag{10}$$

the square of the correlation coefficient, henceforth called the coefficient of determination,

$$\rho_{im}^2 = \frac{\sigma^2(\tilde{R}_{it}) - \sigma^2(\tilde{\epsilon}_{it})}{\sigma^2(\tilde{R}_{it})} = \frac{\beta_i^2 \sigma^2(\tilde{R}_{mt})}{\sigma^2(\tilde{R}_{it})}, \quad t = 1, \ldots, T, \tag{11}$$

is the proportion of the variance of $\tilde{R}_{it}$ that can be attributed to the market—that is, to the term $\beta_i \tilde{R}_{mt}$ in the market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ of equation (3)—while $1 - \rho_{im}^2$ is the fraction of $\sigma^2(\tilde{R}_{it})$ that can be attributed to $\tilde{\epsilon}_{it}$, the error or disturbance of the market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$.

Since the estimation techniques of Chapter 3 are based on the assumption that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is the same for each month of the sampling period, the assumption is maintained in this chapter. We indicate this in the preceding statement of the properties of the market model by appending the notation $t = 1, \ldots, T$ to each equation. Since all properties of the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ are constant or stationary during the sampling period, there is no need for a subscript t on any parameters. We

make use of this prerogative in writing α_i, β_i, and ρ_{im} without t subscripts, but in other cases the prerogative goes unused.

B. Market Model Estimates for IBM

The estimators of the market model coefficients β_i and α_i involve substituting unbiased estimators of $E(\tilde{R}_{it})$, $E(\tilde{R}_{mt})$, $\sigma^2(\tilde{R}_{mt})$, and $\text{cov}(\tilde{R}_{it}, \tilde{R}_{mt})$ into (2). The unbiased estimators of these parameters are

$$\tilde{\tilde{R}}_i = \frac{\sum_{t=1}^{T} \tilde{R}_{it}}{T} \quad \text{and} \quad \tilde{\tilde{R}}_m = \frac{\sum_{t=1}^{T} \tilde{R}_{mt}}{T} \tag{12}$$

$$s^2(\tilde{R}_m) = \frac{\sum_{t=1}^{T} (\tilde{R}_{mt} - \tilde{\tilde{R}}_m)^2}{T - 1} \tag{13}$$

$$\tilde{s}_{im} = \frac{\sum_{t=1}^{T} (\tilde{R}_{it} - \tilde{\tilde{R}}_i)(\tilde{R}_{mt} - \tilde{\tilde{R}}_m)}{T - 1}, \tag{14}$$

so that the estimators of β_i and α_i are

$$\tilde{b}_i = \frac{\tilde{s}_{im}}{s^2(\tilde{R}_m)} = \frac{\sum_{t=1}^{T} (\tilde{R}_{it} - \tilde{\tilde{R}}_i)(\tilde{R}_{mt} - \tilde{\tilde{R}}_m)}{\sum_{t=1}^{T} (\tilde{R}_{mt} - \tilde{\tilde{R}}_m)^2} \tag{15}$$

$$\tilde{a}_i = \tilde{\tilde{R}}_i - \tilde{b}_i \tilde{\tilde{R}}_m. \tag{16}$$

Recall that techniques or procedures for estimating parameters, like those described in equations (12) to (16), are called estimators. When such techniques are applied to particular samples of data, the numbers that they produce are called estimates. An estimator is a random variable, which we indicate with the usual tilde. An estimate is a drawing from the sampling distribution of the estimator, so that when an estimate is referred to, the tilde is dropped. The reader should check that these words and notation are used consistently in what follows.

Suppose now that the common stock i of interest is the common stock of IBM, and we wish to estimate β_i and α_i from the monthly returns on IBM and the equally weighted market portfolio m for the five-year period from July 1963 through June 1968. In this chapter, m includes only NYSE common stocks. The monthly returns, R_{it} and R_{mt}, are shown in Table 4.1. From equation (15) we can see that to estimate β_i and α_i, we must first compute

TABLE 4.1

Monthly Returns, R_{it}, on IBM and on R_{mt}, the Equally Weighted Version of the Market Portfolio, for the Period July 1963–June 1968

MONTH	R_{it}	R_{mt}	MONTH	R_{it}	R_{mt}
7/63	−.0040	−.0095	1/66	−.0060	.0435
8/63	.0259	.0506	2/66	.0413	.0109
9/63	.0163	−.0184	3/66	.0019	−.0219
10/63	.0929	.0163	4/66	.0804	.0337
11/63	−.0152	−.0068	5/66	−.0220	−.0724
12/63	.0448	.0075	6/66	−.0296	−.0048
1/64	.0690	.0201	7/66	−.0278	−.0127
2/64	.0521	.0270	8/66	−.0562	−.0931
3/64	.0444	.0314	9/66	−.0094	−.0143
4/64	−.0404	−.0031	10/66	.0457	.0127
5/64	.0549	.0116	11/66	.1358	.0382
6/64	−.0063	.0154	12/66	−.0120	.0162
7/64	−.0314	.0277	1/67	.0754	.1428
8/64	−.0438	−.0090	2/67	.0791	.0209
9/64	−.0091	.0370	3/67	.0488	.0520
10/64	−.0378	.0170	4/67	.1009	.0365
11/64	−.0149	.0007	5/67	−.0352	−.0179
12/64	−.0073	−.0069	6/67	.0670	.0516
1/65	.0952	.0587	7/67	.0206	.0709
2/65	.0195	.0278	8/67	−.0136	.0028
3/65	−.0033	.0053	9/67	.0970	.0378
4/65	.0677	.0359	10/67	.0825	−.0359
5/65	−.0113	−.0079	11/67	.0330	.0067
6/65	−.0418	−.0743	12/67	.0245	.0554
7/65	.0459	.0291	1/68	−.0518	−.0035
8/65	.0449	.0451	2/68	−.0222	−.0416
9/65	.0271	.0308	3/68	.0560	−.0045
10/65	.0400	.0474	4/68	.1061	.1164
11/65	−.0122	.0300	5/68	.0558	.0586
12/65	−.0495	.0327	6/68	−.0091	.0192

the sample means $\overline{R}_i$ and $\overline{R}_m$. Applying (12) to the returns in Table 4.1 (and the reader may find it instructive to check the calculations that follow), we get

$$\overline{R}_i = \frac{1.2694}{60} = .0212, \quad \overline{R}_m = \frac{.9739}{60} = .0162.$$

Thus, the average monthly return on IBM is 2.12 percent, while the return on the market portfolio is 1.62 percent per month. During this period the shareholders of IBM did quite well, but the market portfolio also had a substantial average monthly return. The sample mean returns, the returns in Table 4.1, and equations (13) to (16) can now be used to compute

$$s^2(R_m) = \frac{.089421}{59} = .001516$$

$$s_{im} = \frac{.060315}{59} = .001022$$

$$b_i = \frac{.06031}{.08942} = .6745$$

$$a_i = .0212 - .6745(.0162) = .0103.$$

Thus, corresponding to the regression function of equation (1), we have the estimated regression function

$$\hat{R}_{it} = a_i + b_i R_{mt}.$$
$$= .0103 + .6745 R_{mt}.$$

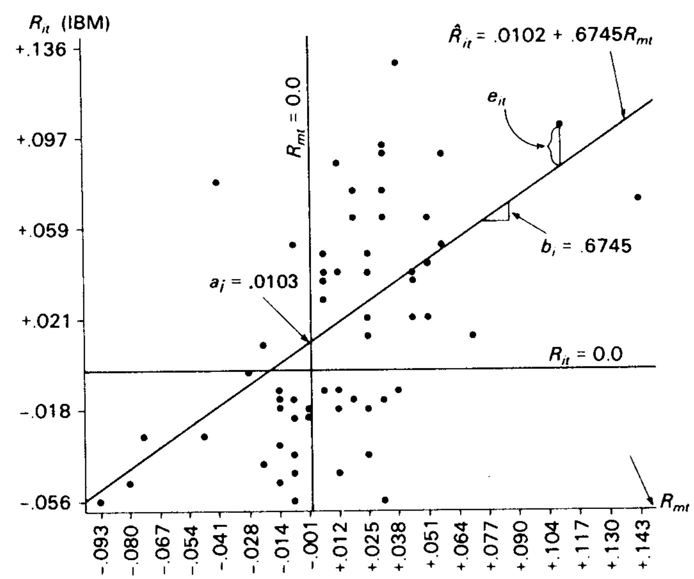
Corresponding to equation (3), we have the estimated market model equation

$$R_{it} = a_i + b_i R_{mt} + e_{it}$$
$$= .0103 + .6745 R_{mt} + e_{it}.$$

The results of the computations are perhaps best appreciated from Figure 4.1, which presents a plot of the sample points (the sample paired values

FIGURE 4.1

Plot of Sample Points and Estimated Market Model Regression Function for IBM for July 1963–June 1968



of R_{it} and R_{mt}), indicated by stars, and of the estimated regression function. The slope of the line is b_i, and a_i is the point on the line where $R_m = 0.0$. The residual e_{it} for any sample point is the vertical distance from the dot corresponding to that sample point to the point on the estimated regression function along the (imaginary) line from the dot that is perpendicular to the R_m axis. One such residual is indicated in the figure.

C. *The Fit of the Estimated Regression*

The impression given by Figure 4.1 is that there is a relationship between the monthly returns on a share of IBM and the monthly returns on the market portfolio m, and the relationship appears to be linear. But it does not seem to be strong, since the dispersion of the sample points around the estimated regression function is substantial. There are several ways to give formal content to this visual impression.

First, from the equation (10), the disturbance variance $\sigma^2(\tilde{\epsilon}_{it})$ measures that part of the variance of the return on security i that cannot be attributed to the market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$. The unbiased estimator of $\sigma^2(\tilde{\epsilon}_{it})$ is

$$s^2(\tilde{e}_i) = \frac{\sum_{t=1}^{T} \tilde{e}_{it}^2}{T-2}. \tag{17}$$

Applying this estimator to the residuals for IBM we get

$$s^2(e_i) = \frac{.09164}{58} = .00158.$$

Using the unbiased estimator of $\sigma^2(\tilde{R}_{it})$,

$$s^2(\tilde{R}_i) = \frac{\sum_{t=1}^{T} (\tilde{R}_{it} - \bar{\tilde{R}}_i)^2}{T-1}, \tag{18}$$

we can also determine that

$$s^2(R_i) = \frac{.13260}{59} = .00225.$$

Since

$$\frac{s^2(e_i)}{s^2(R_i)} = \frac{.00158}{.00225} = .702,$$

the sample estimate is that slightly more than 70 percent of the variance of $\tilde{R}_{it}$ is unexplained by the market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$. Conversely, since

$$1.0 - \frac{s^2(e_i)}{s^2(R_i)} = \frac{s^2(R_i) - s^2(e_i)}{s^2(R_i)} = .298, \tag{19}$$

slightly less than 30 percent of the sample variance of $\tilde{R}_{it}$ can be attributed to the estimated market model relationships between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$.

There is another approach to the same question, and it gives a slightly different answer. Just as equation (10) says that the variance of $\tilde{R}_{it}$ has two components, so we know from Section III of Chapter 3 that

$$\sum_{t=1}^{T} (\tilde{R}_{it} - \bar{\tilde{R}}_i)^2 = \tilde{b}_i^2 \sum_{t=1}^{T} (\tilde{R}_{mt} - \bar{\tilde{R}})^2 + \sum_{t=1}^{T} \tilde{e}_{it}^2; \tag{20}$$

that is, in any sample the sum of squared deviations of $\tilde{R}_{it}$ from its sample mean can be split into "explained" and residual sums of squares. Moreover, from Section III of Chapter 3 we also know that the sample coefficient of determination (the square of the sample correlation coefficient), which is defined as

$$\tilde{r}_{im}^2 = \left(\frac{\tilde{s}_{im}}{s(\tilde{R}_i)s(\tilde{R}_m)}\right)^2, \tag{21}$$

can be expressed as

$$\tilde{r}_{im}^2 = \frac{\tilde{b}_i^2 \sum_{t=1}^{T} (\tilde{R}_{mt} - \bar{\tilde{R}}_m)^2}{\sum_{t=1}^{T} (\tilde{R}_{it} - \bar{\tilde{R}}_i)^2} = \frac{\tilde{b}_i^2 s^2(\tilde{R}_m)}{s^2(\tilde{R}_i)}. \tag{22}$$

Thus, $\tilde{r}_{im}^2$ can be interpreted as the fraction of the sample variance of $\tilde{R}_{it}$ that can be attributed to the fitted market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$.

When equation (21) or (22) is applied to the monthly returns on a share of IBM, the sample coefficient of determination is

$$r_{im}^2 = .307.$$

Thus, slightly more than 30 percent of the sample variance of $\tilde{R}_{it}$ can be attributed to the estimated market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$.

Note that (19) gives a slightly lower measure of the strength of the relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ than (21) or (22), even though the equations purport to measure the same thing. Indeed, (19) and (22) are just the sample counterparts of the two versions of the population coefficient of determination given in equation (11).

Although they are closely related, (19) and (22) are not identical sample quantities. The reason is that although equations (10) and (11) hold for the population variances and although (20) holds for the sample sums of squares, nevertheless

$$s^2(\tilde{R}_i) < \tilde{b}_i^2 s^2(\tilde{R}_m) + s^2(\tilde{e}_i). \tag{23}$$

To see this, simply note that the estimators $s^2(\tilde{R}_i)$ and $s^2(\tilde{R}_m)$ involve dividing the sample sums of squares $\Sigma(\tilde{R}_{it} - \bar{\tilde{R}}_i)^2$ and $\Sigma(\tilde{R}_{mt} - \bar{\tilde{R}}_m)^2$ by $T - 1$, whereas in $s^2(\tilde{e}_i)$ the sum of squared residuals $\Sigma\tilde{e}_{it}^2$ is divided by $T - 2$. Note also that if we define

$$\tilde{r}_{im}^{*2} = \frac{s^2(\tilde{R}_i) - s^2(\tilde{e}_i)}{s^2(\tilde{R}_i)} = 1.0 - \frac{s^2(\tilde{e}_i)}{s^2(\tilde{R}_i)}, \tag{24}$$

then it follows from (23) that

$$\tilde{r}_{im}^2 = \frac{\tilde{b}_i^2 s^2(\tilde{R}_m)}{s^2(\tilde{R}_i)} > \frac{s^2(\tilde{R}_i) - s^2(\tilde{e})}{s^2(\tilde{R}_i)} = \tilde{r}_{im}^{*2}. \tag{25}$$

Because $\tilde{r}_{im}^{*2}$ takes account of the fact that the unbiased estimator $s^2(\tilde{e}_i)$ has fewer degrees of freedom than the unbiased estimators $s^2(\tilde{R}_i)$ and $s^2(\tilde{R}_m)$, $\tilde{r}_{im}^{*2}$ is usually called the sample coefficient of determination, adjusted for degrees of freedom, whereas $\tilde{r}_{im}^2$ is called the sample coefficient of determination. In applications of the market model, the difference between these two measures of fit is usually negligible. Since the sample size T is generally large, a correction involving one degree of freedom has a trivial effect on the estimator.

For example, for IBM, the choice between $r_{im}^{*2} = .298$ and $r_{im}^2 = .307$ is of no consequence. In either case only about 30 percent of the sample variance of the stock's return can be attributed to the estimated market model relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$. Thus, the impression given by Figure 4.1—that the estimated regression function leaves much of the variation in the sample points unexplained—receives formal confirmation.

D. *The Reliability of the Market Model Coefficient Estimates for IBM*

The estimate of β_i for IBM from the monthly returns for July 1963-June 1968 is .67. From Chapters 2 and 3 we know that β_i can be interpreted as the risk of security i in the market portfolio m measured relative to $\sigma^2(\tilde{R}_{mt})$, the risk of m, which is also the average risk of all the securities in m. Since m includes all the common stocks on the NYSE, the estimate $b_i = .67$ suggests that the risk of a share of IBM is substantially less than the average risk in m of all stocks on the exchange. Alternatively, if one interprets β_i as the market sensitivity of security i, then the estimate $b = .67$ suggests that the return on a share of IBM has substantially less than average sensitivity to marketwide factors.

An estimate like $b_i = .67$ is, however, just a drawing from the probability distribution of possible values of the estimator $\tilde{b}_i$ of equation (15). To draw any conclusions from a specific estimate, one must measure its reliability. The first step is to compute the sample estimate of the variance of $\tilde{b}_i$. From Chapter 3, the variance of the estimator $\tilde{b}_i$, conditional on $R_{m1}, \ldots, R_{mT}$, is

$$\sigma^2(\tilde{b}_i | R_{m1}, \ldots, R_{mT}) = \frac{\sigma^2(\tilde{\epsilon}_{it})}{\sum_{t=1}^{T}(R_{mt} - \bar{R}_m)^2} = \frac{\sigma^2(\tilde{\epsilon}_{it})}{(T-1)s^2(R_m)}. \tag{26}$$

The variance of the estimator depends on the strength of the relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$, as measured by the disturbance variance $\sigma^2(\tilde{\epsilon}_{it})$; the weaker the relationship—that is, the larger the value of $\sigma^2(\tilde{\epsilon}_{it})$—the larger the conditional variance of the estimator. The variance of the estimator also depends on the sample size; the larger the value of T, the smaller the conditional variance of $\tilde{b}_i$. Analogous statements apply to the sample estimator of the conditional variance,

$$s^2(\tilde{b}_i | R_{m1}, \ldots, R_{mT}) = \frac{s^2(\tilde{e}_i)}{\sum_{t=1}^{T}(R_{mt} - \bar{R}_m)^2} = \frac{s^2(\tilde{e}_i)}{(T-1)s^2(R_m)}, \tag{27}$$

where $s^2(\tilde{e}_i)$ is the estimator of $\sigma^2(\tilde{\epsilon}_i)$ given by (17).

For IBM the estimate of the conditional variance of $\tilde{b}_i$ for July 1963-June 1968 is

$$s^2(b_i | R_{m1}, \ldots, R_{mT}) = \frac{.00158}{.08942} = .0177,$$

so that

$$s(b_i | R_{m1}, \ldots, R_{mT}) = .1331.$$

This number seems to say that there is substantial uncertainty about the value of β_i for IBM, but let us try to give more formal content to this impression.

Recalling the discussion in Section V.C of Chapter 3, from the Bayesian viewpoint, the uncertainty about β_i that cannot be resolved by the sample at hand is summarized by the posterior distribution on $\tilde{\beta}_i$. For a large sample and given a diffuse prior distribution on $\tilde{\beta}_i$, the posterior distribution on $\tilde{\beta}_i$, conditional on the sample values $R_{m1}, \ldots, R_{mT}$, is approximately normal, with mean

$$E(\tilde{\beta}_i | R_{m1}, \ldots, R_{mT}) = b_i = .6745$$

and standard deviation

$$\sigma(\tilde{\beta}_i | R_{m1}, \ldots, R_{mT}) = s(b_i | R_{m1}, \ldots, R_{mT}) = .1331.$$

If we standardize $\tilde{\beta}_i$ as

$$\tilde{t} = \frac{\tilde{\beta}_i - E(\tilde{\beta}_i | R_{m1}, \ldots, R_{mT})}{\sigma(\tilde{\beta}_i | R_{m1}, \ldots, R_{mT})} = \frac{\tilde{\beta}_i - b_i}{s(b_i | R_{m1}, \ldots, R_{mT})}, \tag{28}$$

then we can use the unit normal distribution tabulated in Table 1.8 to compute some fractiles of the posterior distribution and so get a more concrete feeling for the uncertainty about the value of β_i that the sample does not resolve. For example, the .025 and .975 fractiles of the posterior distribution on $\tilde{\beta}_i$ (corresponding to $t_{.025} = -1.96$ and $t_{.975} = 1.96$) are $\beta_i = .414$ and $\beta_i = .935$. Likewise:

Some Fractiles of the Posterior Distribution of $\tilde{\beta}_i$ for IBM

Cumulative probability	.025	.05	.10	.25	.50	.75	.90	.95	.975
Fractile	.414	.455	.504	.585	.674	.764	.845	.893	.935

These fractiles suggest that there is substantial remaining uncertainty about the value of β_i. The posterior probability is .25 that β_i is less than .585, and the probability is .25 that β_i is greater than .764. Thus the probability is .5 that β_i is outside the interval from .585 to .764. Alternatively, the Bayesian 50 percent confidence interval on β_i is from .585 to .764; that is, the posterior probability that β_i is in this interval is .5. Likewise, the interval from .504 to .845 covers a fairly wide range of possible values of β_i, but the probability that the true β_i is outside this interval is .2, so that the interval is the 80 percent Bayesian confidence interval for β_i. If one prefers the classical to the Bayesian approach to measuring reliability, the fractiles of the Bayesian posterior distribution shown above are nevertheless relevant, since sample estimates of Bayesian and classical confidence intervals are identical.

We soon see that the results for IBM are typical. With samples of five years of monthly returns, there is always substantial uncertainty about the values of β_i for individual common stocks.

E. *Testing the Assumptions Underlying the Coefficient Estimators*

With either the classical or the Bayesian approach, there are two major assumptions from which the properties of the market model regression coefficient estimators derive. The first assumption is that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is bivariate normal. The second assumption is that there is random sampling from the stationary joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$. The purpose of this section is to describe how one might judge the validity of these two assumptions. This is an important task. The validity of the inferences from estimates of parameters depends on whether the assumptions that underlie the statistical techniques used are a good approximation to the data at hand. It is always well to check that this is true.

THE IMPLICATIONS OF BIVARIATE NORMALITY

The assumption that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is bivariate normal has three major implications that are the basis of the market model and of the properties of the market model coefficient estimators. First, bivariate normality implies that the regression function $E(\tilde{R}_{it} | R_{mt})$ is a linear function of R_{mt}. Second, the market model disturbance $\tilde{\epsilon}_{it}$ has a normal distribution, as do the returns $\tilde{R}_{it}$ and $\tilde{R}_{mt}$. Third, the expected value of $\tilde{\epsilon}_{it}$ is zero, and $\tilde{\epsilon}_{it}$ is independent of $\tilde{R}_{mt}$; that is, the conditional distribution of $\tilde{\epsilon}_{it}$ is the same for all values of R_{mt}. The first and third implications are summarized in equations (1) to (6).

Using the sample results for IBM for July 1963-June 1968, let us examine first the implication of bivariate normality that the distributions of $\tilde{R}_{it}$, $\tilde{R}_{mt}$, and $\tilde{\epsilon}_{it}$ are normal. We rely on the studentized range introduced and used extensively in Chapter 1. Recall that the studentized range is the difference between the largest and smallest of the sample values of a random variable, divided by the sample standard deviation. From the sample results for July 1963-June 1968, we get

Studentized Ranges

R_{it}(IBM)	R_{mt}	e_{it}
4.05	6.06	4.56

From Table 1.9 we determine that the sample studentized range (*SR*) for R_{mt}, 6.06, is between the .99 and .995 fractiles of the distribution of the studentized range in samples of size 60 from a normal population. Thus, in sampling from a normal population, there is less than a 1 percent chance that a sample will yield a studentized range as large as or larger than 6.06. On the other hand, the studentized range for the monthly returns on IBM is 4.05, which is between the .05 and .10 fractiles of the distribution of the studentized range, while the studentized range of the residuals is 4.56, which is almost midway between the .10 and .90 fractiles. The range of the returns on m is unusually large for a sample from a normal population; the range of the returns on IBM is slightly small for a sample from a normal population, while the range of the market model residuals is not at all unusual for a sample from a normal population. One might conclude that the sample returns for R_{it} and e_{it} are consistent with normality but those for R_{mt} are not.

From one viewpoint, nonnormality of $\tilde{R}_{mt}$ is not critical. Recall from Chapter 3 that the distributional properties of the market model coefficient estimators do not require that the distribution of $\tilde{R}_{mt}$ be normal. The critical assumptions are random sampling and normality for the disturbances $\tilde{\epsilon}_{it}$, and the studentized range of IBM's residuals is consistent with the hypothesis of normality for $\tilde{\epsilon}_{it}$. From the viewpoint of the two-parameter portfolio model, however, which is based on the assumption that all portfolio return distributions are normal, it is disturbing if the market portfolio m, which is representative of a diversified portfolio, has a return distribution that is substantially nonnormal.

There is the possibility that the extreme studentized range for R_{mt} for July 1963-June 1968 is due to chance, so that the distribution of $\tilde{R}_{mt}$ is not so nonnormal as this five-year period might suggest. To check this possibility, we examine the studentized ranges for R_{mt} for various subperiods from February 1926 to June 1968:

Period	2/26-12/30	1/31-12/35	1/36-12/40	1/41-12/45	1/46-12/50
$SR(R_m)$	4.75	5.29	5.94	4.42	4.46
T	59	60	60	60	60
Period	**1/51-12/55**	**1/56-12/60**	**1/61-12/65**	**1/66-6/68**	
$SR(R_m)$	4.42	5.12	5.78	4.83	
T	60	60	60	30	

Two of these studentized ranges, those for 1/36-12/40 and 1/61-12/65, are extreme in the sense that they exceed the .975 fractile of the relevant sampling distribution of SR in Table 1.9; and two, those for 1/31-12/35 and 1/66-6/68, are also extreme (but less so) inasmuch as they exceed the .90 fractile of the relevant sampling distributions of SR. On the other hand, the studentized ranges for the remaining five periods are quite consistent with what would be expected from normal populations. Three of them, those for

the three five-year periods 1941-1955, might even be said to fall slightly into the left tail of the relevant sampling distribution of SR.

In short, the results are consistent with a distribution for $\tilde{R}_{mt}$ that is slightly leptokurtic relative to a normal distribution, but much less nonnormal than one might infer from the one studentized range for July 1963-June 1968. This is, of course, similar to the conclusion that we draw with respect to distributions of monthly returns on securities and portfolios in Chapter 1, where frequency distributions and studentized ranges of monthly returns are studied in detail, and where we conclude that for monthly returns normal distributions are a workable approximation.

Next we consider the implications of bivariate normality that there is a linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ and that the disturbances $\tilde{\epsilon}_{it}$ from this linear relationship are independent of $\tilde{R}_{mt}$, so that the conditions of equations (1) to (6) hold. Perhaps the best—or, at least in practice, the most common—way to judge the validity of these propositions is by inspection of a plot of the sample combinations of R_{it} and R_{mt} like that shown in Figure 4.1, with the estimated regression function also included on the graph. Obviously, visual inspection can only lead to impressionistic judgments about the validity of the propositions.

Thus, to judge the validity of the proposition that the regression function $E(\tilde{R}_{it}|R_{mt})$ is a linear function of R_{mt}, which is what we mean when we say that there is a linear relationship between the two variables, we can visually inspect a graph like Figure 4.1 and judge whether some nonlinear function might provide a better fit to the sample points. If, as in Figure 4.1, a linear function seems appropriate, then we can conclude that linearity of the regression function is an appropriate approximation. This allows us to conclude that the proposition of equation (4), that the conditional expected value of $\tilde{\epsilon}_{it}$ is independent of R_{mt}, is also an appropriate approximation to the data; that is, if the regression function $E(\tilde{R}_{it}|R_{mt})$ is a linear function of R_{mt}, then $E(\tilde{\epsilon}_{it}|R_{mt})$ must be zero for all values of R_{mt}.

A graph like Figure 4.1 can also be used to judge the validity of the statement of equation (5) that the variance of $\tilde{\epsilon}_{it}$ is independent of R_{mt}, but a combination of care and artistry is needed. In terms of its implications for a sample, equation (5) says that the dispersion of the sample points about the estimated regression function should be about the same for different values of R_{mt}. But one must be careful in interpreting the word "dispersion." Extreme values of R_{mt} are, after all, much less likely than values close to the mean of R_{mt}. Thus, in a sample there are likely to be fewer drawings from the distribution of $\tilde{\epsilon}_{it}$ corresponding to extreme values of R_{mt} than there are for more moderate values of R_{mt}. As a consequence, even though $\sigma^2(\tilde{\epsilon}_{it}|R_{mt})$ may be the same for all values of R_{mt}, more extreme observations on $\tilde{e}_{it}$ should be more numerous the closer one is to .016, the mean of R_{mt}, since the expected frequency of sample points is higher for intervals closer to the mean of R_{mt}. The range of sample points about the estimated regression function should, however, be about the same at points that are equal distances to the left or right of the mean of R_{mt}, and extreme deviations of sample points from the estimated regression function should be less numerous the further one looks away from the mean of R_{mt}. To my eye, the results for IBM shown in Figure 4.1 are consistent with these qualitative statements.

A more formal approach to examining the proposition that $\sigma^2(\tilde{e}_{it}|R_{mt})$ is the same for all values of R_{mt} is to divide the sample range of R_{mt} into intervals that contain the same number of sample observations and then to compute the standard deviations of the residuals in each interval. If the proposition that $\sigma^2(\tilde{e}_{it}|R_{mt})$ is the same for all R_{mt} is valid, these sample standard deviations should be approximately the same for all the intervals of R_{mt}. Note that, according to the comments of the preceding paragraph, if each of the intervals of R_{mt} contains the same number of sample points, then the closer the interval is to $\bar{R}_m$, the smaller the range of values of R_{mt} it will cover.

Finally, recall from Section III.C of Chapter 3 that the sample mean residual cannot be used to test the proposition of equation (4) that the unconditional expected value of the disturbances $E(\tilde{e}_{it}) = 0$, since the coefficient estimates b_i and a_i are defined in such a way that in any sample

$$\bar{e}_i = \frac{\sum_{t=1}^{T} e_{it}}{T} = 0.$$

Moreover, it is also always true that in any sample,

$$\sum_{t=1}^{T} (R_{mt} - \bar{R}_m)\, e_{it} = 0,$$

so that the sample covariance between R_{mt} and e_{it} cannot be used to test the proposition of equation (6) that cov $(\tilde{e}_{it}, \tilde{R}_{mt}) = 0.0$.

STATIONARITY AND RANDOM SAMPLING

Having examined the implications of bivariate normality, we now turn to the questions of whether the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is constant or stationary during the sampling period, and whether successive paired values of the monthly returns can be regarded as independent drawings from the joint distribution of the returns.

Time series plots of returns and residuals. Figures 4.2 to 4.4 present plots against time of the monthly IBM returns, the monthly returns on the market portfolio m, and the monthly residuals from the estimated market model regression function. Such time series plots allow us to judge whether the distributions of the returns and residuals remain constant during the sampling period. Specifically, the time series plots are excellent for judging whether

FIGURE 4.2

Time Series Plot of Monthly Returns on IBM for July 1963–June 1968

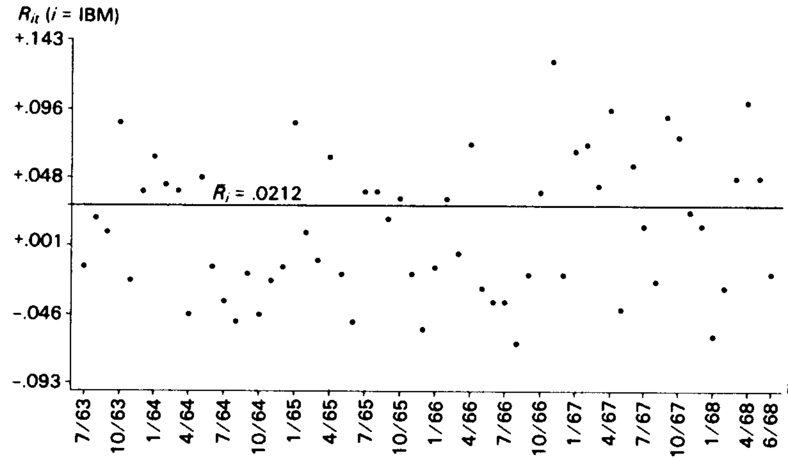


FIGURE 4.3

Times Series Plot of Monthly Returns on the Market Portfolio m *for July 1963–June 1968*

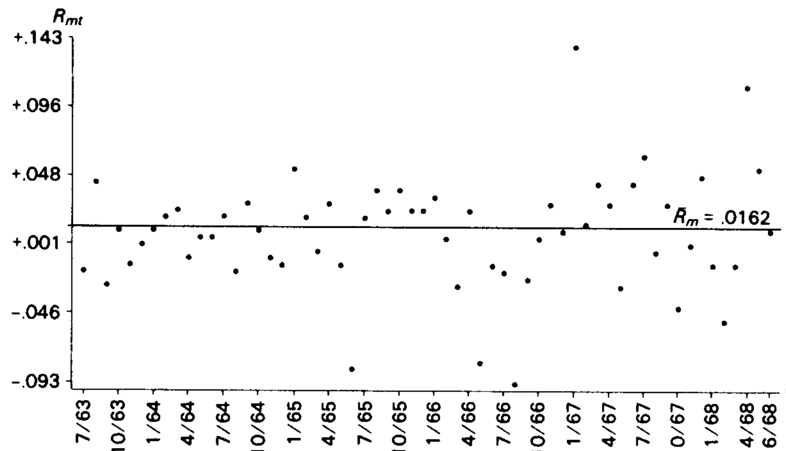
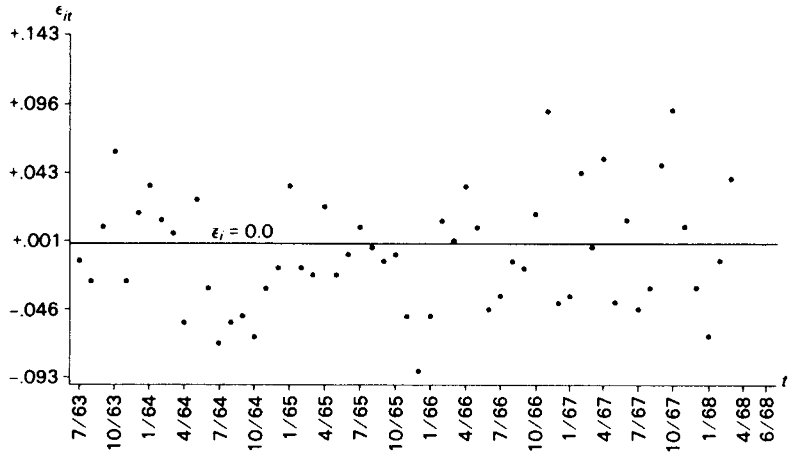


FIGURE 4.4

Time Series Plot of the Market Model Residuals for IBM for July 1963–June 1968



the variances of the variables change through time, and this is an important question. For example, the assumption that the disturbance variance $\sigma^2(\tilde{\epsilon}_{it})$ is constant through time is used to derive the expressions for the conditional variances of the market model coefficient estimators. The plot of the residuals in Figure 4.4 does not seem to raise serious doubts about this assumption. The plots of R_{it} and R_{mt} likewise do not suggest any obvious changes during the sampling period in the behavior of the monthly returns on IBM and on the market portfolio m.

With plots like Figures 4.2 to 4.4, however, one can only judge changes in the separate behavior of the returns and residuals, whereas the assumption is that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is stationary during the sampling period. This assumption indeed implies that the distributions of returns and residuals are stationary, but it also implies that the market model regression coefficients α_i and β_i are constant during the sampling period. We discuss this proposition later.

Autocorrelations of returns and residuals. To some extent, Figures 4.2 to 4.4 can also be used to judge the assumption of random sampling. With random sampling from the bivariate distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$, successive values of $\tilde{R}_{it}$ are independent, as are successive values of $\tilde{R}_{mt}$ and $\tilde{\epsilon}_{it}$. In terms of Figures 4.2 to 4.4, this means that there should not be runs of higher than average or lower than average returns or residuals, above and beyond the runs that would be expected by chance. Equivalently, through time the returns and residuals should be randomly scattered about their respective means.*

Although the plots always provide valuable insights and familiarity with the properties of the data, some amount of bunching through time of high or low returns or residuals is to be expected on a purely chance basis. This makes visual inspection of the behavior of the variables a tricky procedure for judging whether any patterns observed are consistent or inconsistent with the assumption of random sampling. However, quantitative procedures for testing the assumption of random sampling are also available. Fortunately, these procedures are based on statistical concepts that we have already studied.

Let us illustrate the approach in terms of the returns $\tilde{R}_{it}$. The goal is to measure the relationship between the returns $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$, that is, returns τ months apart. Suppose we are willing to limit attention to a possible linear regression function relationship of the form

$$E(\tilde{R}_{it} | R_{i,t-\tau}) = \delta_\tau + \gamma_\tau R_{i,t-\tau} \qquad (29)$$

so that the return can be expressed as

$$\tilde{R}_{it} = \delta_\tau + \gamma_\tau \tilde{R}_{i,t-\tau} + \tilde{\xi}_{it}. \qquad (30)$$

Finally, we assume that the process generating the returns is stationary through time; that is, the process is the same for all t, so that, as indicated in (29), the coefficients δ_τ and γ_τ in the relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ are the same for all t.

From Section II of Chapter 3, we know that if δ_τ and γ_τ are defined in the usual way as

$$\gamma_\tau = \frac{\operatorname{cov}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})}{\sigma^2(\tilde{R}_{i,t-\tau})} \text{ and } \delta_\tau = E(\tilde{R}_{it}) - \gamma_\tau E(\tilde{R}_{i,t-\tau}), \qquad (31)$$

then

$$\operatorname{cov}(\tilde{\xi}_{it}, \tilde{R}_{i,t-\tau}) = 0.0, \qquad (32)$$

so that

$$\sigma^2(\tilde{R}_{it}) = \gamma_\tau^2 \sigma^2(\tilde{R}_{i,t-\tau}) + \sigma^2(\tilde{\xi}_{it}). \qquad (33)$$

*The statistical properties of the market model coefficient estimators discussed in Chapter 3 require random sampling from the distribution of the disturbances $\tilde{\epsilon}_{it}$. But when the properties of the estimators are not based on assumed bivariate normality for $\tilde{R}_{it}$ and $\tilde{R}_{mt}$, random sampling from the distributions of $\tilde{R}_{mt}$, and $\tilde{R}_{it}$ is not necessary. We find in Chapter 5, however, that independence through time of security and portfolio returns is an important issue in its own right. Thus, it helps to set up our later work if we now discuss testing for time series independence both for the returns $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ and for the disturbances $\tilde{\epsilon}_{it}$.

It then follows that if we define the correlation coefficient between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ as

$$\rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = \frac{\operatorname{cov}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})}{\sigma(\tilde{R}_{it})\sigma(\tilde{R}_{i,t-\tau})}, \tag{34}$$

then the coefficient of determination

$$\rho^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = \left(\frac{\operatorname{cov}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})}{\sigma(\tilde{R}_{it})\sigma(\tilde{R}_{i,t-\tau})}\right)^2 = \frac{\gamma_\tau^2 \sigma^2(\tilde{R}_{i,t-\tau})}{\sigma^2(\tilde{R}_{it})} \tag{35}$$

is the proportion of the variance of $\tilde{R}_{it}$ that can be attributed to the linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$—that is, to the term $\gamma_\tau \tilde{R}_{i,t-\tau}$ in (30)—while $1 - \rho^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ is $\sigma^2(\tilde{\xi}_{it})/\sigma^2(\tilde{R}_{it})$, the proportion of the variance of $\tilde{R}_{it}$ that can be attributed to the disturbance $\tilde{\xi}_{it}$ in (30).

The correlation coefficient between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ is given a special name: it is called the autocorrelation for lag τ. It is also sometimes called the serial correlation for lag τ. The assumption that the statistical process is the same for all t implies that the standard deviation of $\tilde{R}_{it}$ is the same for all t,

$$\sigma(\tilde{R}_{it}) = \sigma(\tilde{R}_{i,t-\tau}) = \sigma(\tilde{R}_i). \tag{36}$$

It follows from (36) that the autocorrelation for lag τ is also the linear regression coefficient γ_τ. Under the latter interpretation, it is called the autoregression coefficient for lag τ.

The sample estimators of δ_τ, γ_τ, and $\rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ are defined in the usual way: we simply plug in sample estimators of the covariance, the means, and the standard deviations that appear in (31) and (34). We have

$$\tilde{g}_\tau = \frac{s(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})}{s^2(\tilde{R}_{i,t-\tau})} = \frac{\sum_{t=\tau+1}^{T} (\tilde{R}_{it} - \tilde{\tilde{R}}_{it})(\tilde{R}_{i,t-\tau} - \tilde{\tilde{R}}_{i,t-\tau})}{\sum_{t=\tau+1}^{T} (\tilde{R}_{i,t-\tau} - \tilde{\tilde{R}}_{i,t-\tau})^2} \tag{37}$$

$$\tilde{d}_\tau = \tilde{\tilde{R}}_{it} - \tilde{g}_\tau \tilde{\tilde{R}}_{i,t-\tau} \tag{38}$$

$$\tilde{\tilde{R}}_{it} = \frac{\sum_{t=\tau+1}^{T} \tilde{R}_{it}}{T-\tau}, \tag{39}$$

$$\tilde{\tilde{R}}_{i,t-\tau} = \frac{\sum_{t=\tau+1}^{T} \tilde{R}_{i,t-\tau}}{T-\tau}. \tag{40}$$

To see the logic in these expressions, note that a sample of T observations on $\tilde{R}_{it}$ only yields $T-\tau$ paired observations on $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ that can be used to estimate the coefficients of (30). That is, the sample points, the paired values of R_{it} and $R_{i,t-\tau}$, that can be used to estimate the coefficients are the $T-\tau$ pairs

$$(R_{i,\tau+1}, R_{i1}), (R_{i,\tau+2}, R_{i2}), \ldots, (R_{iT}, R_{i,T-\tau}).$$

The estimators of the sample means, variances, and covariances then simply reflect the fact that the sample observations on $\tilde{R}_{it}$ are $R_{i,\tau+1}, \ldots, R_{iT}$, while the sample observations on $R_{i,t-\tau}$ are $R_{i1}, \ldots, R_{i,T-\tau}$.

One consequence of this way of looking at the sample is that although the assumed stationarity of the process implies

$$E(\tilde{R}_{it}) = E(\tilde{R}_{i,t-\tau}) = E(\tilde{R}_i),$$

this equality does not hold for the sample estimators; that is,

$$\tilde{\tilde{R}}_{it} \neq \tilde{\tilde{R}}_{i,t-\tau}.$$

Likewise, although stationarity implies (36), nevertheless in any sample

$$s^2(\tilde{R}_{it}) \neq s^2(\tilde{R}_{i,t-\tau}).$$

Finally, if the sample estimator of the autocorrelation coefficient for lag τ is defined as

$$\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = \frac{s(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})}{s(\tilde{R}_{it})s(\tilde{R}_{i,t-\tau})} = \frac{\sum_{t=\tau+1}^{T} (\tilde{R}_{it} - \tilde{\tilde{R}}_{it})(\tilde{R}_{i,t-\tau} - \tilde{\tilde{R}}_{i,t-\tau})}{\sqrt{\sum_{t=\tau+1}^{T} (\tilde{R}_{it} - \tilde{\tilde{R}}_{it})^2}\ \sqrt{\sum_{t=\tau+1}^{T} (\tilde{R}_{i,t-\tau} - \tilde{\tilde{R}}_{i,t-\tau})^2}}, \tag{41}$$

then

$$\tilde{g}_\tau \neq \tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau});$$

that is, although $\gamma_\tau = \rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$, the sample estimators of the two quantities are not the same.

This last result seems to cause a problem, since either $\tilde{g}_\tau$ or $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ can be regarded as an estimator of the autocorrelation for lag τ. In practice, however, as long as the sample period T is long—more specifically, as long as τ is small relative to T—then $\tilde{g}_\tau$ and $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ will be nearly identical.

The property of the sample autocorrelation that we use most in later discussions is the fact that, like any sample coefficient of determination, $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})^2$ can be shown to be

$$\tilde{r}^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = \frac{\tilde{g}_\tau^2 \sum_{t=\tau+1}^{T} (\tilde{R}_{i,t-\tau} - \tilde{\bar{R}}_{i,t-\tau})^2}{\sum_{t=\tau+1}^{T} (\tilde{R}_{it} - \tilde{\bar{R}}_{it})^2}, \tag{42}$$

which can always be interpreted as the proportion of the sample sum of squares

$$\sum_{t=\tau+1}^{T} (\tilde{R}_{it} - \tilde{\bar{R}}_{it})^2$$

that can be attributed to the estimated linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$. Alternatively, $\tilde{r}^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ is an estimator of $\rho^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$, which, from (35), can be interpreted as the proportion of the variance of $\tilde{R}_{it}$ that can be attributed to the linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$. In applications, we commonly rely on this "proportion of variance explained" interpretation of $r^2(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ as an indication of the degree of dependence between values of $\tilde{R}_{it}$ that are τ months apart. If the proportion of variance explained by the linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ is close to zero, then we conclude that the assumption that returns which are separated by τ months are independent is a reasonable approximation to the data.

As usual, we generally want to judge the reliability of $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ as an estimator of $\rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$. Like any other estimator, $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ is a random variable with a sampling distribution; as always, we want to know how tightly concentrated the distribution is about the true value of the parameter of interest. The analysis of this problem is in general quite difficult, but fortunately there are some simple results for the case of most interest. In the applications of this and later chapters, we are almost always interested in the distribution of $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ under the hypothesis that $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ are independent, so that $\rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = 0.0$. When $\rho(\tilde{R}_{it}, \tilde{R}_{i,t-\tau}) = 0.0$, the distribution of $\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ in large samples is approximately normal, with mean and standard deviation

$$E[\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})] \doteq -\frac{1}{T-\tau} \tag{43}$$

$$\sigma[\tilde{r}(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})] \doteq \sqrt{\frac{1}{T-\tau}}. \tag{44}$$

An individual sample autocorrelation $r(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ allows us to judge whether returns τ months apart are independent. The discussion above applies to any τ, however, so we can compute the values of $r(\tilde{R}_{it}, \tilde{R}_{i,t-\tau})$ for $\tau = 1$, $\tau = 2$, etc., and use these to judge the degree of dependence between values of $\tilde{R}_{it}$ that are separated by one month, by two months, and so forth. If there does not seem to be an important amount of dependence for any τ, then we can conclude that the assumption of random sampling—that is, the assumption that successive values of $\tilde{R}_{it}$ are independent—is a reasonable approximation to the data.

Moreover, although for purposes of illustration the preceding discussion has been carried out in terms of the returns $\tilde{R}_{it}$, the analysis and results apply to any random variable that can be regarded as a time series. Thus, they can be used now to help us decide whether the assumption of random sampling is a reasonable approximation for the monthly returns on IBM, the monthly returns on the market portfolio m, and the market model residuals for IBM for July 1963-June 1968. Table 4.2 shows the sample autocorrelations of

TABLE 4.2

Autocorrelation Estimates for the Monthly Returns on IBM, the Market Portfolio m, and the Market Model Residuals for IBM for July 1963-June 1968

LAG τ	$R_{it}(i = \text{IBM})$	R_{mt}	e_{it}	$\sigma(r_\tau)$
1	.139	.111	.213	.130
2	.022	.013	−.071	.131
3	−.003	.103	−.114	.132

the three variables for lags $\tau = 1, 2, 3$. The standard deviations of the coefficients, computed from (44), are also shown. For example, the table says that the sample correlation between values of the return on IBM one month apart is .14. Thus, we estimate that approximately $(.14)^2 \cong .02$, or only 2 percent, of the variance of $\tilde{R}_{it}$ for IBM can be attributed to the linear relationship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-1}$, which is consistent with the proposition that $\tilde{R}_{it}$ and $\tilde{R}_{i,t-1}$ are independent. Indeed, all of the sample autocorrelations shown in Table 4.2 are in this sense small, as are the coefficients for lags greater than 3, which are not shown. We conclude that the assumption of random sampling is consistent with the data.

We close by noting that since the interpretation of an autocorrelation is linked to a linear regression function relationship like (29), the autocorrelation is a measure of linear dependence. A linear relationship is just one possible form for the relationship between lagged values of a random variable. In practice, however, autocorrelations are the primary tool used to measure serial dependence.

PROBLEM I.E.

1. Below are the monthly returns on a share of Xerox common stock for July 1963-June 1968. For convenience, the returns on the market portfolio m are also shown. Fit the market model to these data. Specifically, compute b_i, a_i, $s(b_i|R_{m1}, \ldots, R_{mT})$, $s(a_i|R_{m1}, \ldots, R_{mT})$, r_{im}^2, the studentized ranges of R_{it} and e_{it}, and the autocorrelations of R_{it} and e_{it} for lags $\tau = 1, 2, 3$. The estimating equation for $s(a_i|R_{m1}, \ldots, R_{mT})$ is (49) of Chapter 3.

MONTH (t)	R_{it}	R_{mt}	MONTH (t)	R_{it}	R_{mt}
7/63	.2471	-.0095	1/66	.0755	.0435
8/63	.1653	.0506	2/66	.0834	.0109
9/63	-.0075	-.0164	3/66	.0454	-.0219
10/63	.2954	.0163	4/66	.0269	.0337
11/63	.0274	-.0068	5/66	-.0406	-.0724
12/63	.1389	.0075	6/66	.0155	-.0046
1/64	-.0772	.0201	7/66	-.0738	-.0127
2/64	.0095	.0270	8/66	-.2191	-.0931
3/64	.0784	.0314	9/66	-.0148	-.0143
4/64	.1141	-.0031	10/66	-.0736	.0127
5/64	.2228	.0116	11/66	.2670	.0382
6/64	-.0013	.0154	12/66	-.0366	.0162
7/64	-.0935	.0277	1/67	.1703	.1428
8/64	-.0420	-.0090	2/67	.0784	.0209
9/64	.2213	.0370	3/67	.1250	.0320
10/64	-.1221	.0170	4/67	.0228	.0365
11/64	-.1168	.0007	5/67	-.0506	-.0179
12/64	.0450	-.0069	6/67	.0055	.0516
1/65	.1255	.0567	7/67	-.0183	.0709
2/65	.1239	.0278	8/67	-.0173	.0028
3/65	-.0311	.0053	9/67	.0551	.0378
4/65	.1253	.0359	10/67	.0531	-.0359
5/65	.0791	-.0079	11/67	-.0022	.0067
6/65	-.0358	-.0743	12/67	.0395	.0554
7/65	.0947	.0291	1/68	-.1650	-.0035
8/65	.1022	.0451	2/68	-.0253	-.0416
9/65	-.0088	.0308	3/68	-.0183	-.0045
10/65	.0356	.0474	4/68	.1489	.1164
11/65	.1219	.0300	5/68	.0904	.0586
12/65	.0313	.0327	6/68	-.0182	.0192

ANSWER

1. The values of the coefficients, their standard deviations, and so forth are shown in Tables 4.3 and 4.5.

II. Evidence on the Risks or Market Sensitivities of NYSE Common Stocks

A. *Comments on Market Model Estimates for Larger and Smaller Firms*

Table 4.3 shows the market model coefficient estimates b_i and a_i, computed from monthly returns for July 1963-June 1968, for the thirty common stocks that account for the largest fractions of the total market value of outstanding shares on the NYSE at the end of 1971. The estimates of the conditional standard deviations of the coefficients are also shown. Henceforth we refer to these as the standard deviations or standard errors of the coefficients; that is, we no longer explicitly include the word conditional, and the estimates are denoted as $s(b_i)$ and $s(a_i)$. For each of the stocks, Table 4.3 also shows the sample standard deviation of the market model residuals, $s(e_i)$; the sample coefficient of determination, r_{im}^2; and the sample mean and standard deviation, $\overline{R}_i$ and $s(R_i)$, of the stock's return. Table 4.4 shows the corresponding results for a random sample of NYSE stocks.

The first thing to note is that in the results for the larger firms in Table 4.3, only two of the b_i (Xerox and Ford) are greater than 1.0, and most are substantially less than 1.0. The average of the b_i in Table 4.3 is only .61. Interpreting β_i as the risk of security i measured relative to the average risk of securities in m, the estimates imply that the risks of the common stocks of larger firms tend to be substantially less than the average risk of stocks in m. Alternatively, interpreting β_i as the sensitivity of the return on security i to marketwide factors, the larger stocks seem to have less than average market sensitivity. In contrast, in Table 4.4, 15 of the randomly selected stocks have $b_i > 1.0$, 15 have $b_i < 1.0$, and the average of the b_i is 1.00. Thus, these stocks do not tend to have either systematically more or less risk than the average risk in m of all common stocks on the exchange. This is, of course, exactly what we expect from a random sample.

The second point to note from Tables 4.3 and 4.4 is that there seems to be a relationship between $s(R_i)$ and b_i. Generally, the larger the value of b_i, the larger the sample standard deviation of the security's returns. This result has two causes, one algebraic and one that is just an empirical finding. First, the sample sum of squares from which $s(R_i)$ is computed can be expressed as

$$\sum_{t=1}^{T} (R_{it} - \overline{R}_i)^2 = b_i^2 \sum_{t=1}^{T} (R_{mt} - \overline{R}_m)^2 + \sum_{t=1}^{T} e_{it}^2. \qquad (45)$$

122

TABLE 4.3

Market Model Parameter Estimates for 30 Largest Firms for July 1963–June 1968

NAME	% OF 1971 MARKET VALUE	b_i	$s(b_i)$	a_i	$s(a_i)$	r^2_{im}	$\bar{R}_i$	$s(R_i)$	$s(e_i)$
International Business Machines	5.473	.67	.133	.010	.0056	.31	.0212	.047	.040
American Telephone and Telegraph	3.477	.19	.110	−.002	.0046	.05	.0008	.033	.033
General Motors	3.275	.87	.145	−.006	.0061	.39	.0081	.055	.043
Exxon	2.336	.24	.121	.000	.0051	.06	.0043	.037	.036
Sears, Roebuck	2.244	.45	.150	.003	.0063	.13	.0105	.048	.045
Eastman Kodak	2.222	.65	.147	.011	.0062	.25	.0215	.050	.044
General Electric	1.609	.78	.187	−.007	.0079	.23	.0055	.063	.056
Xerox	1.389	1.20	.310	.021	.0130	.20	.0400	.103	.093
Texaco	1.324	.43	.134	−.001	.0056	.15	.0059	.043	.040
Ford Motor	1.087	1.04	.151	−.012	.0063	.45	.0054	.061	.045
Minneapolis Mining and Manufacturing	1.073	.79	.158	.001	.0066	.30	.0133	.056	.047
Coca-Cola	1.027	.38	.147	.016	.0062	.10	.0221	.046	.044
Du Pont	.969	.65	.134	−.009	.0056	.29	.0013	.047	.040
Procter and Gamble	.907	.30	.129	.001	.0054	.08	.0063	.040	.038
Gulf Oil	.848	.24	.137	.009	.0057	.05	.0124	.042	.041
Mobil Oil	.784	.27	.161	.005	.0068	.05	.0095	.049	.048
Johnson and Johnson	.777	.91	.176	.007	.0074	.31	.0222	.063	.053
Standard Oil (California)	.688	.35	.126	−.001	.0053	.12	.0046	.040	.038
Standard Oil (Indiana)	.679	.42	.184	.007	.0077	.08	.0140	.057	.055
Royal Dutch Petroleum	.659	.62	.180	−.000	.0075	.17	.0099	.059	.054
Shell Transport and Trading	.649	.55	.178	.008	.0075	.14	.0173	.057	.053
American Home Products	.639	.83	.161	.003	.0068	.32	.0166	.058	.048
Merck and Company	.639	.60	.159	.011	.0066	.20	.0212	.053	.047
International Telephone and Telegraph	.594	.99	.160	.001	.0067	.40	.0168	.061	.048
J. C. Penney	.578	.89	.177	.000	.0074	.30	.0145	.063	.053
Westinghouse Electric	.539	.99	.207	.001	.0087	.28	.0169	.072	.062
Dow Chemical	.533	.70	.173	−.003	.0072	.22	.0085	.058	.052
Kresge, S. S.	.513	.49	.220	.033	.0092	.08	.0410	.068	.066
General Telephone and Electronics	.497	.48	.136	.004	.0057	.17	.0116	.044	.040
Atlantic Richfield	.460	.40	.149	.014	.0063	.11	.0203	.047	.045
Averages	1.282	.61	.161	.004	.0068	.20	.0141	.054	.048

123

TABLE 4.4

Market Model Coefficient Estimates for 30 Randomly Selected Firms for July 1963–June 1968

NAME	b_i	$s(b_i)$	a_i	$s(a_i)$	r^2_{im}	$\bar{R}_i$	$s(R_i)$	$s(e_i)$
IPL	1.20	.340	−.013	.0142	.18	.0070	.111	.102
Lehigh Portland Cement	1.07	.207	−.014	.0087	.32	.0034	.074	.062
Hotel Corporation of America	1.60	.338	.009	.0142	.28	.0356	.118	.101
Portec	1.51	.272	−.005	.0114	.35	.0196	.100	.081
Richardson Merrill	.69	.245	.002	.0103	.12	.0128	.078	.073
Van Raalte	.73	.179	.009	.0075	.22	.0208	.060	.054
Ex-Cell-O	1.11	.191	−.002	.0080	.37	.0157	.071	.057
Keebler	1.14	.202	.002	.0085	.36	.0201	.075	.060
Canadian Breweries	.08	.259	.024	.0108	.00	.0253	.077	.077
Gulf, Mobile and Ohio Railroad	1.30	.201	.004	.0084	.42	.0247	.078	.060
Dana Corporation	.84	.123	−.007	.0051	.45	.0066	.049	.037
Union Pacific Railroad	.66	.140	−.002	.0059	.28	.0086	.050	.042
Cyclops Corporation	.87	.177	−.005	.0074	.30	.0095	.063	.053
Ohio Edison	.28	.128	.002	.0053	.07	.0069	.039	.038
Central Foundry	2.24	.413	−.018	.0173	.34	.0188	.150	.124
United States Gypsum	1.01	.180	−.011	.0076	.35	.0053	.066	.054
Eversharp	1.22	.334	−.017	.0140	.19	.0032	.110	.100
Dayton Power and Light	.58	.145	−.002	.0061	.22	.0072	.049	.043
Cluett, Peabody and Company	.67	.196	.009	.0082	.17	.0197	.064	.059
Washington Gas Light	.14	.028	−.002	.0039	.04	−.0001	.028	.028
Lowenstein, M., and Sons	1.21	.214	−.001	.0089	.36	.0189	.079	.064
International Telephone and Telegraph	.99	.160	.001	.0067	.40	.0168	.061	.048
Carpenter Steel	.83	.227	.006	.0095	.19	.0197	.075	.068
Greyhound	.93	.176	−.008	.0074	.32	.0074	.063	.053
Allegheny Ludlum Steel	.66	.184	.004	.0077	.18	.0142	.060	.055
United Air Lines	1.30	.286	.000	.0120	.26	.0210	.099	.086
Adams Express	.40	.087	.006	.0036	.27	.0122	.030	.026
Ambac Industries	2.03	.265	−.004	.0111	.50	.0287	.111	.079
Masonite	1.39	.224	−.006	.0094	.40	.0170	.086	.067
Lehigh Valley Industries	1.34	.522	.032	.0219	.10	.0533	.163	.156
Averages	1.00	.221	.000	.0093	.27	.0158	.078	.067

Since $\Sigma(R_{mt} - \bar{R}_m)^2$ is the same for every security, we can see that the larger the value of b_i, the larger the value of $s^2(R_i) = \Sigma(R_i - \bar{R})^2/T - 1$. Second a universal empirical finding in the literature is that larger values of b_i tend to be associated with larger values of $s^2(e_i) = \Sigma\, e_{it}^2/(T-2)$. This relationship can be seen in Table 4.3, but it is more evident in Table 4.4, where there are more pronounced differences in the b_i of different securities.

Although both components of $\Sigma(R_{it} - \bar{R}_i)^2$ in (45) tend to increase with b_i, the residual sum of squares, $\Sigma\, e_{it}^2$, must increase less (in percentage terms) than $b_i^2\, \Sigma(R_{mt} - \bar{R}_m)^2$. This conclusion is implied by the observation that the sample coefficients of determination, r_{im}^2, seem also to increase with b_i. For example, the average values of b_i and r_{im}^2 for the stocks in Table 4.3 are .61 and .20, whereas the average values of b_i and r_{im}^2 for the randomly selected stocks in Table 4.4 are 1.00 and .27. Since r_{im}^2 can be written as

$$r_{im}^2 = \frac{b_i^2 \sum_{t=1}^{T} (R_{mt} - \bar{R}_m)^2}{\sum_{t=1}^{T} (R_{it} - \bar{R}_i)^2},$$

a positive relationship between r_{im}^2 and b_i implies that the numerator of this equation, $b_i^2\, \Sigma(R_{mt} - \bar{R}_m)^2$, increases with b_i more (in percentage terms) than the denominator, $\Sigma(R_{it} - \bar{R}_i)^2$, which in turn implies that $\Sigma\, e_{it}^2$ does not increase with b_i as much (in percentage terms) as $b_i^2\, \Sigma(R_{mt} - \bar{R}_m)^2$.

The final point to note from Tables 4.3 and 4.4 is that for July 1963–June 1968, marketwide factors always explain 50 percent or less of the sample variances of the returns on the individual stocks shown in the tables. The sample coefficients of determination r_{im}^2 are all .5 or less.

B. *Evidence on the Assumptions Underlying the Market Model Estimates*

Table 4.5 shows sample statistics that can be used to test the assumptions underlying the market model regression coefficient estimates in Table 4.3. For each stock in Table 4.3, Table 4.5 shows the studentized ranges, $SR(R_i)$ and $SR(e_i)$ of the stock's returns and of its market model residuals for July 1963–June 1968, along with the sample autocorrelations $r(R_{it}, R_{i,t-\tau})$ and $r(e_{it}, e_{i,t-\tau})$, $\tau = 1, 2, 3$. Table 4.6 shows the corresponding studentized ranges and sample autocorrelations for the randomly selected securities in Table 4.4.

Interpreting the squared sample autocorrelations as estimates of the proportion of the variance of $\tilde{R}_{it}$ or $\tilde{e}_{it}$ that can be attributed to a linear rela-

76

125

TABLE 4.5

Studentized Ranges and Autocorrelations of Returns and Residuals for 30 Largest Firms, July 1963–June 1968

COMPANY	$SR(R_i)$	$SR(e_i)$	$r(R_{it}, R_{i,t-1})$	$r(R_{it}, R_{i,t-2})$	$r(R_{it}, R_{i,t-3})$	$r(e_{it}, e_{i,t-1})$	$r(e_{it}, e_{i,t-2})$	$r(e_{it}, e_{i,t-3})$
International Business Machines	4.05^{-}	4.56	.139	.022	-.003	.213	-.071	-.114
American Telephone and Telegraph	5.52^{+}	5.97^{+}	-.111	.096	.173	-.088	.161	.099
General Motors	4.52	5.83^{+}	-.091	-.060	.254	-.013	-.132	.192
Exxon	4.60	4.88	-.025	-.032	.242	.030	-.036	.125
Sears, Roebuck	5.81^{+}	6.27^{+}	-.105	-.020	.253	-.114	-.055	.204
Eastman Kodak	3.98^{-}	3.98^{-}	.098	-.175	.088	.057	-.067	.055
General Electric	5.58^{+}	5.49^{+}	-.028	-.093	-.006	.078	-.119	-.089
Xerox	4.99	4.71	.039	.065	-.067	.063	-.026	-.021
Texaco	5.00	5.02	.076	-.148	.004	.169	-.103	-.113
Ford Motor	4.64	4.17	-.083	-.183	.115	-.067	-.151	-.025
Minnesota Mining and Manufacturing	4.30	4.14	-.062	-.055	-.071	.034	.047	.054
Coca-Cola	4.16	6.06^{+}	-.085	-.027	-.044	-.159	-.041	-.178
Du Pont	4.79	4.56	-.076	-.023	.234	.036	-.039	.038
Procter and Gamble	5.52^{+}	5.15	-.193	.192	-.077	-.162	.269	-.107
Gulf Oil	4.13	4.17	-.019	.016	-.065	.045	.027	-.126
Mobil Oil	4.49	4.24	-.234	.023	-.296	-.190	.059	-.348
Johnson and Johnson	4.32	5.15	-.056	-.056	-.033	-.084	-.109	-.184
Standard Oil (California)	4.74	4.98	-.111	.093	.207	-.127	.074	.098
Standard Oil (Indiana)	4.72	4.71	-.210	-.117	.175	-.056	-.126	.103
Royal Dutch Petroleum	4.81	4.87	-.032	-.321	.271	-.031	-.337	.162
Shell Transport and Trading	4.77	5.00	.171	-.044	.288	.118	-.081	.175
American Home Products	3.97^{-}	5.07	-.080	.108	.115	-.171	.005	.026
Merck and Company	4.34	4.24	-.159	-.065	-.090	-.146	-.044	-.078
International Telephone and Telegraph	4.28	4.79	.005	-.065	-.015	.017	-.222	-.087
J. C. Penney	4.05^{-}	4.23	-.159	-.051	.171	-.088	-.048	.053
Westinghouse Electric	4.32	4.36	.099	-.005	-.094	.114	-.085	-.199
Dow Chemical	4.96	5.43^{+}	-.049	-.046	.100	-.174	-.108	-.013
Kresge, S. S.	4.71	4.57	-.156	.033	-.059	-.209	.029	-.076
General Telephone and Electronics	4.47	4.28	-.022	.000	.083	.053	.098	.065
Atlantic Richfield	4.13	4.08	-.030	-.284	.088	.035	-.287	.025
Averages	4.62	4.83	-.051	-.041	.064	-.027	-.051	-.009

$^{-}$The sample studentized range is less than 4.07, the .1 fractile of the distribution of the studentized range in samples of size 60 from a normal population.
$^{+}$The sample studentized range exceeds 5.29, the .9 fractile of the distribution of the studentized range in samples of size 60 from a normal population.

126

Studentized Ranges and Autocorrelations of Returns and Residuals for 30 Randomly Selected Firms, July 1963–June 1968

COMPANY	$SR(R_i)$	$SR(e_i)$	$r(R_{it}, R_{i,t-1})$	$r(R_{it}, R_{i,t-2})$	$r(R_{it}, R_{i,t-3})$	$r(e_{it}, e_{i,t-1})$	$r(e_{it}, e_{i,t-2})$	$r(e_{it}, e_{i,t-3})$
IPL	5.46+	5.71+	-.050	.114	.009	-.240	.076	.034
Lehigh Portland Cement	6.47+	5.50+	.092	-.047	-.080	.021	-.186	-.096
Hotel Corporation of America	4.73	5.06	.026	.258	-.012	-.061	.181	-.101
Portec	6.19+	5.75+	-.192	.011	.386	-.195	-.066	.213
Richardson Merrill	4.38	5.55+	-.066	.018	-.276	-.157	.083	-.307
Van Raalte	5.20	4.05-	-.093	-.117	.127	-.185	-.106	.029
Ex-Cell-O	5.18	4.37	-.101	.138	.165	-.124	.012	.008
Keebler	4.64	4.34	.322	.218	.038	.124	.292	-.047
Canadian Breweries	5.33+	5.24	-.011	-.140	.348	-.025	-.157	.357
Gulf, Mobile and Ohio Railroad	4.40	3.93-	-.056	.098	.113	-.182	-.056	.171
Dana Corporation	6.19+	4.92	-.053	-.125	.269	-.099	-.161	.109
Union Pacific Railroad	4.67	4.41	.064	.040	.010	.028	-.028	.006
Cyclops Corporation	4.22	4.73	.044	-.010	.038	-.154	.017	.279
Ohio Edison	4.29	4.54	-.142	-.071	-.075	-.162	-.048	-.118
Central Foundry	5.65+	5.91+	-.120	-.003	-.061	-.335	-.039	.029
United States Gypsum	5.59+	5.55+	.004	.169	-.058	-.010	.106	-.056
Eversharp	4.30	4.62	-.028	.054	-.059	-.060	.097	-.085
Dayton Power and Light	4.47	4.87	-.074	-.025	-.116	-.014	-.016	-.156
Cluett, Peabody and Company	5.74+	5.59+	.105	.215	.129	-.049	.246	.034
Washington Gas Light	4.52	4.53	-.121	-.096	.123	-.074	-.127	.037
Lowenstein, M., and Sons	4.22	4.23	-.022	-.159	.342	-.019	-.139	.144
International Telephone and Telegraph	4.29	4.80	.005	-.065	-.015	.017	-.223	-.087
Carpenter Steel	4.99	4.97	-.198	-.016	-.182	-.225	.031	-.175
Greyhound	4.23	4.21	.027	-.025	.175	-.006	.005	.014
Allegheny Ludlum Steel	4.47	4.41	-.203	.027	.092	-.136	.029	.079
United Air Lines	5.23	5.40+	.111	.042	.020	.154	-.012	.199
Adams Express	5.41+	4.71	-.198	.110	.039	-.181	.052	-.090
Ambac Industries	5.34+	5.07	.089	.144	.152	.163	-.069	-.089
Masonite	5.57+	4.34	-.117	-.031	.082	-.109	-.105	.186
Lehigh Valley Industries	6.49+	6.50+	-.288	.166	-.017	-.266	.206	-.014
Averages	5.06	4.92	-.030	-.030	.057	-.085	-.004	.017

+The sample studentized range exceeds 5.29, the .9 fractile of the distribution of the studentized range in samples of size 60 from a normal population.

-The sample studentized range is less than 4.07, the .1 fractile of the distribution of the studentized range in samples of size 60 from a normal population.

tionship between $\tilde{R}_{it}$ and $\tilde{R}_{i,t-\tau}$ or between $\tilde{\epsilon}_{it}$ and $\tilde{\epsilon}_{i,t-\tau}$, the autocorrelation estimates for both R_{it} and e_{it} seem consistent with the assumption that successive values of $\tilde{R}_{it}$ and of $\tilde{\epsilon}_{it}$ are independent. The largest measured autocorrelations are in excess of .3 in absolute value, implying a 9 percent estimated explanation of variance, but most of the measured autocorrelations are much closer to zero. Moreover, when so many autocorrelations for so many different securities are computed, one can expect a few extreme values to be observed on a purely chance basis. Attributing the large measured autocorrelations to chance seems reasonable, since their signs are not systematically positive or negative.

The studentized ranges shown in Table 4.5 are consistent with the hypothesis that the returns and market model disturbances for the large firms are from normal distributions. As would be expected under the hypothesis of normality for $\tilde{R}_{it}$, most of the values of $SR(R_i)$ in Table 4.5 fall into the central portion of the sampling distribution of SR; and of the "extreme" values of $SR(R_i)$, four are less than 4.07, the .10 fractile of the sampling distribution of SR, and four are greater than 5.29, the .9 fractile of the sampling distribution of SR. The average of the $SR(R_i)$ is 4.62, which is just about halfway between the .1 and .9 fractiles of the sampling distribution of SR. Similar comments apply to the studentized ranges $SR(e_i)$ for the market model residuals of the companies in Table 4.5.

A slightly different picture emerges for the randomly selected firms in Table 4.6. For 12 of the 30 firms, the values of $SR(R_i)$ exceed 5.29, the .9 fractile of the sampling distribution of SR in samples of 60 from a normal population. The distributions of returns for these firms show slightly higher frequencies of extreme returns than would be expected under the hypothesis of normality. The studentized ranges for the market model residuals of the firms in Table 4.6 likewise suggest slight leptokurtosis; nine of the $SR(e_i)$ exceed the .90 fractile of the sampling distribution of SR, while only two of the $SR(e_i)$ are less than the .10 fractile of the sampling distribution of SR. Thus, the assumption of normality is a better approximation for the returns of larger firms than for those of randomly selected firms, but even for the latter we shall continue to see how far the normality assumption can take us in our theoretical and empirical work.

It would be well to use plots like Figures 4.1 to 4.4 to check the assumptions that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is bivariate normal and that the return distributions are stationary through time for each of the 60 securities in Tables 4.3 to 4.6. This would, however, consume much space. Suffice it to say that the graphs for IBM are typical. For other common stocks, plots of R_{it} against R_{mt}, like Figure 4.1, seem roughly consistent with the implications of bivariate normality; and time series plots, like Figures 4.2 to 4.4, seem consistent with the assumption that return distributions are stationary, at least for five-year subperiods.

Finally, the essence of the market model is that, to a greater or lesser extent, depending on the value of β_i, the returns on all securities are related to the return on the market portfolio m. That is, the market model equation (3) says that part of the return on any security i for which $\beta_i \neq 0$ is the return on m. Thus, although we have 60 different firms in Tables 4.3 to 4.6, we do not have 60 independent samples of returns. One implication of this is that there can be much interdependence across firms in the sample values of a given statistic. For example, from equation (45) we can determine that as long as b_i is nonzero, the sample variance of the return on the market portfolio is a component of the sample variance of the returns on any common stock. Thus, the sample estimates of return variances for individual firms are interdependent because each depends on the sample variance of the return on the market. Likewise, the values of other sample statistics, such as $SR(R_i)$ and $r(R_{it}, R_{i,t-\tau})$, are interdependent across firms when there are common factors that affect the returns on all firms.

One might suspect that there is little or no dependence across firms in the values of sample statistics, such as $SR(e_i)$ and $r(e_{it}, e_{i,t-\tau})$, which are computed from the market model residuals. However, this is only true if the return on the market portfolio m does a good job in capturing the effects of common factors on the returns of individual firms, so that there is little dependence across firms in the market model disturbances $\tilde{\epsilon}_{it}$. We shall return to this point in Chapter 9.

C. *Comparison of Prewar and Postwar Market Model Parameter Estimates*

In Chapter 1 we found that there is a dramatic downward shift in the variance of the return on the market portfolio m sometime in the late 1930s. We stated there, without evidence, that a similar downward shift in the variances of the returns on individual stocks can also be observed at about the same time. We now present some evidence on this point. We also discuss some interesting changes in the properties of the market model.

Table 4.7 shows estimates of the market model parameters for 1934-1938 for those securities of Table 4.3 that were listed on the NYSE throughout the 1934-1938 period. Table 4.8 reproduces the results in Table 4.4 for those firms in Table 4.4 that were on the NYSE throughout the 1934-1938 period. The decline in the variability of returns on individual securities from 1934-1938 to 1963-1968 is evident. Only one firm, Richardson Merrill, shows a higher value of $s(R_i)$ in the later period than in the earlier period. There is

TABLE 4.7

Market Model Parameter Estimates for Larger Firms for January 1934-December 1938

COMPANY	b_i	$s(b_i)$	a_i	$s(a_i)$	r^2_{im}	$\bar{R}_i$	$s(R_i)$	$s(e_i)$
International Business Machines	.27	.051	.006	.0054	.32	.0112	.050	.041
American Telephone and Telegraph	.33	.039	.005	.0042	.55	.0115	.048	.032
General Motors	.77	.072	.001	.0077	.66	.0154	.100	.059
Exxon	.54	.064	−.001	.0068	.55	.0087	.077	.052
Sears, Roebuck	.73	.076	.005	.0081	.62	.0183	.099	.062
Eastman Kodak	.38	.061	.012	.0065	.40	.0193	.064	.050
General Electric	.72	.062	.007	.0066	.70	.0203	.091	.050
Texaco	.68	.094	.009	.0101	.47	.0211	.105	.077
Coca-Cola	.32	.079	.029	.0084	.22	.0345	.072	.064
Du Pont	.50	.055	.004	.0059	.59	.0133	.069	.045
Procter and Gamble	.38	.070	.005	.0075	.34	.0119	.070	.057
Mobil Oil	.63	.080	−.007	.0086	.52	.0043	.093	.065
Standard Oil (California)	.54	.068	−.009	.0073	.51	.0003	.079	.056
American Home Products	.48	.057	.008	.0061	.56	.0165	.069	.046
International Telephone and Telegraph	.89	.118	−.015	.0126	.49	.0016	.134	.096
J. C. Penney	.57	.068	.005	.0072	.55	.0151	.081	.055
Westinghouse Electric	.86	.082	.013	.0087	.65	.0283	.112	.067
Kresge, S. S.	.53	.073	.006	.0078	.47	.0154	.081	.059
Atlantic Richfield	.63	.066	−.008	.0071	.61	.0033	.086	.054
Averages	.56	.069	.004	.0074	.52	.0142	.082	.057

TABLE 4.8

Market Model Parameter Estimates for Randomly Selected Firms for January 1934–December 1938

COMPANY	b_i	$s(b_i)$	a_i	$s(a_i)$	r^2_{im}	$\bar{R}_i$	$s(R_i)$	$s(e_i)$
Lehigh Portland Cement	1.13	.114	.001	.0122	.63	.0210	.151	.093
Hotel Corporation of America	1.54	.148	.002	.0158	.65	.0305	.203	.121
Portec	1.54	.125	-.001	.0133	.72	.0266	.192	.102
Richardson Merrill	.34	.053	.009	.0057	.41	.0154	.056	.043
Van Raalte	1.01	.115	.026	.0123	.57	.0440	.142	.094
Keebler	.50	.073	-.006	.0078	.44	.0030	.079	.060
Gulf, Mobile and Ohio Railroad	1.57	.161	-.009	.0171	.62	.0197	.211	.131
Dana Corporation	1.18	.157	.008	.0167	.49	.0293	.178	.128
Union Pacific Railroad	.65	.068	-.005	.0072	.61	.0063	.088	.055
United States Gypsum	.79	.108	.010	.0116	.48	.0245	.121	.088
Cluett, Peabody and Company	1.02	.155	.014	.0166	.43	.0326	.165	.126
International Telephone and Telegraph	.89	.118	-.015	.0126	.49	.0016	.134	.096
Allegheny Ludlum Steel	.90	.099	-.000	.0106	.58	.0163	.124	.081
Adams Express	1.22	.085	-.004	.0090	.78	.0184	.146	.069
Ambac Industries	1.43	.171	-.003	.0183	.55	.0234	.205	.139
Lehigh Valley Industries	1.26	.150	-.030	.0160	.55	-.0067	.180	.122
Averages	1.06	.119	.000	.0127	.56	.0191	.135	.093

also a clear-cut decline in residual standard deviations from the earlier to the later period. Given that $s(R_m)$ also declined, we can use these results and equation (45) to conclude that the decline in the variability of a security's return generally reflects a decline both in the variability of marketwide factors, as summarized by $\tilde{R}_{mt}$, and in the variability of the disturbance $\tilde{e}_{it}$.

Perhaps the most interesting evidence is that the decline in the variability of $\tilde{R}_{mt}$ is sharper, in percentage terms, than the typical decline in the variability of $\tilde{e}_{it}$. The evidence on this point is in the substantial decline in the coefficients of determination, r^2_{im}, from the earlier to the later period. On average, marketwide factors account for 56 percent of security return variances in 1934–1938 for the securities in Table 4.8, whereas for July 1963–June 1968, the corresponding average value of r^2_{im} in Table 4.4 is only .27. Likewise, in the later period the average value of r^2_{im} for the stocks of larger firms in Table 4.3 is .20, as compared to .52 for the earlier period (Table 4.7). Thus, a much smaller fraction of the variance of the return on a security can typically be attributed to its market model relationship with $\tilde{R}_{mt}$ for July 1963–June 1968 than for 1934–1938.

The decline in the explanatory power of the market model was first documented by King (1966); Blume (1968) later documented the declines in r^2_{im}, $s^2(R_i)$, $s^2(R_m)$, and $s^2(e_i)$ in more detail and suggested that the declines are best interpreted as a shift that took place sometime around 1940. Finally, Officer (1971) corroborated Blume's results and investigated several possible reasons for the decline in r^2_{im}. None of the explanations turned out to be supported convincingly by the evidence.

D. *The Reliability of the Risk Estimates*

In discussing the detailed results for IBM for July 1963–June 1968, we concluded that the sample estimate $b_i = .67$ left substantial uncertainty with respect to the value of β_i. The same conclusion holds for the other common stocks we have examined. Thus, from the Bayesian viewpoint, the uncertainty that remains about β_i after a sample has been analyzed is summarized by the posterior distribution on the parameter. With a diffuse prior, a large sample, and under the assumption that the joint distribution of $\tilde{R}_{it}$ and $\tilde{R}_{mt}$ is bivariate normal, the posterior distribution on the parameter is approximately normal, with mean $E(\tilde{\beta}_i) = b_i$ and standard deviation $\sigma(\tilde{\beta}_i) = s(b_i)$. The values of b_i and of $s(b_i)$ for July 1963–June 1968 for each of the stocks in the two samples discussed above are in Tables 4.3 and 4.4. The impression is the same as for IBM. The values of $s(b_i)$ are large, so that the sample estimates leave substantial uncertainty about the values of β_i for the individual stocks. We leave it to the reader to buttress this impression by computing some